

Fractional Operators in the Matrix Variate Case

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[This paper is dedicated to Professor Dr. Francesco Mainardi at his 70th birthday]

Abstract

Fractional integral operators connected with real-valued scalar functions of matrix argument are applied in problems of mathematics, statistics and natural sciences. In this article we start considering the case of a Gauss hypergeometric function with the argument being a rectangular matrix. Subsequently some fractional integral operators are introduced which complement these results available on fractional operators in the matrix variate cases. Several properties and limiting forms are derived. Then the pathway idea is incorporated to move among several different functional forms. When these are used as models for problems in the natural sciences then these can cover the ideal situations, neighborhoods, in between stages and paths leading to optimal situations.

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1. Introduction

The importance of anomalous reaction/relaxation and transport/diffusion is well recognized in many disciplines including physics, chemistry, biology, and engineering. Despite this fact, anomalous relaxation and transport are not well understood, and there is the need to develop mathematical and statistical models with predictive power. Recent developments of fractional calculus in terms of integro-differential operators that provide a unifying framework to model key aspects of anomalous relaxation and transport like non-locality, non-Markovian (memory) effects, and non-Gaussian (Levy) processes became available (cf. Mathai et al. [11], Nair [4]). The extension of such developments to matrix-variate statistical densities in general (Mathai and Haubold [8], Mainardi [13]) and to the specifically interesting case of Mittag-Leffler functions and matrix-variate analogues (Mathai [7]) have been achieved more recently. Such results are applicable to the solution of linear coupled fractional differential equations (Lim et al. [1]) as well as to the handling of fractional Poisson probability distributions (Laskin [12]). Further use of fractional integral operators connected with real-valued scalar functions of matrix argument can be utilized for fractional matrix calculus (Phillips [3]), numerical solution of fractional diffusion-wave equations (Garg and Manohar [6]), stability analysis of fractional-order systems (Jiao and Chen [9]), and probably for the application of the matrix-variate Mellin transform in radar image processing (Anfinson and Eltoft [10]).

Let $X = (x_{ij})$ be a $p \times r$, $r \geq p$ matrix of real distinct scalar variables x_{ij} 's. Let A be a $p \times p$ real positive definite constant matrix, that is, $A = A' > O$, prime denoting the transpose. Let B be a constant

positive definite $r \times r$ matrix, $B > O$. Let $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ denote the positive definite square roots of A and B respectively. Let X be of full rank p . Then $Z_X = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}$ is symmetric positive definite matrix. In this paper we will consider only real matrices. The corresponding results in the complex domain can be done parallel to the real case. The following standard notations will be used. Real matrix variate gamma will be denoted by

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}) \dots \Gamma(\alpha - \frac{j-1}{2}), \quad (1.1)$$

for $\Re(\alpha) > \frac{p-1}{2}$ where $\Re(\cdot)$ denotes the real part of (\cdot) . It can be shown that $\Gamma_p(\alpha)$ has the integral representation

$$\Gamma_p(\alpha) = \int_{S>O} |S|^{\alpha-\frac{p+1}{2}} e^{-\text{tr}(S)} dS, \quad \Re(\alpha) > \frac{p-1}{2} \quad (1.2)$$

where $|S|$ means the determinant of the $p \times p$ positive definite matrix S and $\text{tr}(S)$ denotes the trace of S . The wedge product of the $\frac{p(p+1)}{2}$ differentials dx_{ij} 's will be denoted by

$$dS = \prod_{i \geq j} \wedge ds_{ij}, \quad \wedge = \text{wedge} \quad (1.4)$$

The type-1 beta integral is given by

$$B_p(\alpha, \beta) = \frac{\Gamma_p(\alpha) \Gamma_p(\beta)}{\Gamma_p(\alpha + \beta)} = \int_{O < X < I} |S|^{\alpha-\frac{p+1}{2}} |I-S|^{\beta-\frac{p+1}{2}} dS, \quad (1.5)$$

for $\Re(\alpha) > \frac{p-1}{2}$, $\Re(\beta) > \frac{p-1}{2}$, where S is $p \times p$ positive definite and $O < S < I$ means $S > O$, $I-S > O$ or all the eigenvalues of S are in the open interval $(0, 1)$. In general, \int_X means the integral over X . The type-2 beta integral is given by

$$\begin{aligned} B_p(\alpha, \beta) &= \int_{S>O} |S|^{\alpha-\frac{p+1}{2}} |I+S|^{-(\alpha+\beta)} dS \\ &= \int_{U>O} |U|^{\beta-\frac{p+1}{2}} |I+U|^{-(\alpha+\beta)} dU, \quad \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2} \end{aligned} \quad (1.6)$$

Let

$$Z_X = A^{\frac{1}{2}}XBX'A^{\frac{1}{2}} \text{ and } Z_Y = A^{\frac{1}{2}}YBY'A^{\frac{1}{2}}$$

where $X = (x_{ij})$ and $Y = (y_{ij})$ are $p \times r$, $r \geq p$ matrices of real elements, and of full rank p . Consider the evaluation of the integral

$$I_1 = \int_X |Z_X|^a |I-Z_X|^{c-a-\frac{p+1}{2}} |I-Z_Y Z_X|^{-b} dX, \quad (1.7)$$

for $O < Z_X < I$, $O < Z_Y < I$. This integral corresponds to the Euler integral for Gauss hypergeometric function. Let

$$U = A^{\frac{1}{2}}XB^{\frac{1}{2}} \Rightarrow dU = |A|^{\frac{r}{2}} |B|^{\frac{r}{2}} dX \quad (\text{see Mathai, 1997, equation (1.2.20)}) \quad (1.8)$$

Then, after integration over the Stiefel manifold,

$$Z_X = UU' = V \Rightarrow dU = \frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\frac{r}{2})} |V|^{\frac{r}{2}-\frac{p+1}{2}} dV \quad (1.9)$$

(see Mathai, 1997, Theorem 2.16 and Remark 2.13). Now, the integral I_1 becomes,

$$\begin{aligned} I_1 &= |A|^{-\frac{r}{2}} |B|^{-\frac{r}{2}} \frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\frac{r}{2})} \\ &\times \int_{O < V < I} |V|^{a+\frac{r}{2}-\frac{p+1}{2}} |I-V|^{c-a-\frac{p+1}{2}} |I-Z_Y^{\frac{1}{2}} V Z_Y^{\frac{1}{2}}|^{-b} dV. \end{aligned} \quad (1.10)$$

Let us expand the last factor in the integrand in terms of zonal polynomials. For a discussion of zonal polynomials see Mathai et al. (1995).

$$|I - Z_Y^{\frac{1}{2}} V Z_Y^{\frac{1}{2}}|^{-b} = \sum_{k=0}^{\infty} \sum_K \frac{(b)_K}{k!} C_K(Z_Y V) \quad (1.11)$$

where $C_K(\cdot)$ is the zonal polynomial of order k , $K = (k_1, k_2, \dots, k_p)$, $k_1 + \dots + k_p = k$ and

$$(b)_K = \prod_{j=1}^p (b - \frac{j-1}{2})_{k_j} = \frac{\Gamma_p(b, K)}{\Gamma_p(b)},$$

$$\Gamma_p(b, K) = \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^p \Gamma(b + k_j - \frac{j-1}{2}) = \Gamma_p(b)(b)_K \quad (1.12)$$

and $(a)_m$ is the Pochhammer symbol

$$(a)_m = a(a+1)\dots(a+m-1), (a)_0 = 1, a \neq 0.$$

We can evaluate the integral

$$\int_{O < V < I} |V|^{a+\frac{r}{2}-\frac{p+1}{2}} |I-V|^{c-a-\frac{p+1}{2}} C_K(Z_Y V) dV = \frac{\Gamma_p(a+\frac{r}{2}, K) \Gamma_p(c-a)}{\Gamma_p(c+\frac{r}{2}, K)} C_K(Z_Y) \quad (1.13)$$

(see Mathai, 1997, (5.1.26)). Note that interchange of integrals and sums is valid here. Then

$$I_1 = |A|^{-\frac{r}{2}} |B|^{-\frac{p}{2}} \frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\frac{r}{2})} \frac{\Gamma_p(a+\frac{r}{2}) \Gamma_p(c-a)}{\Gamma_p(c+\frac{r}{2})}$$

$$\times \sum_{k=0}^{\infty} \sum_K \frac{(b)_K (a+\frac{r}{2})_K}{(c+\frac{r}{2})_K} \frac{C_K(Z_Y)}{k!}. \quad (1.14)$$

2. Hypergeometric Functions of Rectangular Matrix Argument

In the notations of hypergeometric function of matrix argument the series part of (1.14) can be written as a ${}_2F_1$. That is,

$$\sum_{k=0}^{\infty} \sum_K \frac{(b)_K (a+\frac{r}{2})_K}{(c+\frac{r}{2})_K} \frac{C_K(Z_Y)}{k!} = {}_2F_1(a+\frac{r}{2}, b; c+\frac{r}{2}; Z_Y), \quad \|Z_Y\| < 1 \quad (2.1)$$

where $\|(\cdot)\|$ denotes a norm of (\cdot) . Hence we have the following theorem:

Theorem 2.1. For X, Y, A, B as defined in Section 1

$${}_2F_1(a+\frac{r}{2}, b; c+\frac{r}{2}; Z_Y) = \frac{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\frac{r}{2}) \Gamma_p(c+\frac{r}{2})}{\pi^{\frac{rp}{2}} \Gamma_p(a+\frac{r}{2}) \Gamma_p(c-a)}$$

$$\times \int_X |Z_X|^a |I - Z_X|^{c-a-\frac{p+1}{2}} |I - Z_Y Z_X|^{-b} dX \quad (2.2)$$

for $O < Z_X < I, O < Z_Y < I, \Re(c-a) > \frac{p-1}{2}, \Re(a) > -\frac{r}{2} + \frac{p-1}{2}$.

For establishing some limiting forms and pathways we need the following results, which will be stated as lemmas.

Lemma 2.1. For $K = (k_1, \dots, k_p)$, $k_1 + \dots + k_p = k$, $(a)_K = \prod_{j=1}^p (a - \frac{j-1}{2})_{k_j}$,

$$\lim_{q \rightarrow 1} (q-1)^k \left(\frac{1}{q-1} \right)_K = 1.$$

Note that

$$\begin{aligned} (q-1)^k \left(\frac{1}{q-1} \right)_K &= \left\{ \prod_{j=1}^p (q-1)^{k_j} \right\} \left\{ \prod_{i=1}^{k_j} \left(\frac{1}{q-1} - \left(\frac{j-1}{2} \right) + i - 1 \right) \right\} \\ &= \prod_{j=1}^p [1 - (q-1) \left(\frac{j-1}{2} \right) + (q-1)(i-1)] = 1. \end{aligned}$$

Lemma 2.2.

$$\lim_{q \rightarrow 1} |I + (q-1)Z_Y|^{-\frac{1}{q-1}} = e^{-\text{tr}(Z_Y)}. \quad (2.4)$$

Proof: Let λ_j , $j = 1, \dots, p$ be the eigenvalues of Z_Y . Then

$$|I + (q-1)Z_Y| = \prod_{j=1}^p (1 + (q-1)\lambda_j).$$

But

$$\begin{aligned} \prod_{j=1}^p [\lim_{q \rightarrow 1} \{1 + (q-1)\lambda_j\}^{-\frac{1}{q-1}}] &= \prod_{j=1}^p e^{-\lambda_j} = e^{-\sum_{j=1}^p \lambda_j} \\ &= e^{-\text{tr}(Z_Y)}. \end{aligned} \quad (2.5)$$

3. Fractional Integral Operators

Let Z_X and Z_Y be as defined in Section 1. Let the generalized fractional integral operator of matrix argument be defined and denoted as

$$\begin{aligned} ({}_0D_X^{-\alpha} f)(X) &= \frac{1}{\Gamma_p(\alpha)} \int_{Z_X > Z_Y > O} |Z_X - Z_Y|^{\alpha - \frac{p+1}{2}} f(Z_Y) dY \\ &= \frac{|Z_X|^{\alpha - \frac{p+1}{2}}}{\Gamma_p(\alpha)} \int_{Z_X > Z_Y > O} |I - Z_X^{-\frac{1}{2}} Z_Y Z_X^{-\frac{1}{2}}|^{\alpha - \frac{p+1}{2}} f(Z_Y) dY. \end{aligned}$$

Make the transformations $U = A^{\frac{1}{2}} Y B^{\frac{1}{2}}$, $V = U U'$, $W = Z_X^{-\frac{1}{2}} V Z_X^{-\frac{1}{2}}$. Then integrating out over the Stiefel manifold we have

$$\begin{aligned} {}_0D_X^{-\alpha} f &= \frac{|Z_X|^{\alpha + \frac{r}{2} - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\alpha) \Gamma_p(\frac{r}{2})} \\ &\quad \times \int_{O < W < I} |I - W|^{\alpha - \frac{p+1}{2}} f(Z_X^{\frac{1}{2}} W Z_X^{\frac{1}{2}}) |W|^{\frac{r}{2} - \frac{p+1}{2}} dW. \end{aligned} \quad (3.1)$$

Consider the special cases of $f(\cdot)$. Consider the operator operating on a power function.

Case 1. Let $f(Z_X) = |Z_X|^\eta$. Then

$$\begin{aligned} {}_0D_X^{-\alpha} |Z_X|^\eta &= \frac{|Z_X|^{\alpha + \frac{r}{2} + \eta - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\alpha) \Gamma_p(\frac{r}{2})} \int_{O < W < I} |W|^{\frac{r}{2} + \eta - \frac{p+1}{2}} |I - W|^{\alpha - \frac{p+1}{2}} dW \\ &= \frac{|Z_X|^{\alpha + \frac{r}{2} + \eta - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\frac{r}{2})} \frac{\Gamma_p(\frac{r}{2} + \eta)}{\Gamma_p(\alpha + \frac{r}{2} + \eta)}, \Re(\frac{r}{2} + \eta) > \frac{p-1}{2}. \end{aligned} \quad (3.2)$$

Case 2. Zonal polynomial of order k , $f(Z_X) = C_K(Z_X)$

Then going through the same steps as above

$$\begin{aligned} {}_0D_X^{-\alpha} C_K(Z_X) &= \frac{|Z_X|^{\alpha + \frac{r}{2} - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\alpha) \Gamma_p(\frac{r}{2})} \int_{O < W < I} |W|^{\frac{r}{2} - \frac{p+1}{2}} \\ &\quad \times |I - W|^{\alpha - \frac{p+1}{2}} C_K(Z_X^{\frac{1}{2}} W Z_X^{\frac{1}{2}}) dW, \text{ (Mathai, 1997, equation (5.1.26))} \\ &= \frac{|Z_X|^{\alpha + \frac{r}{2} - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\alpha) \Gamma_p(\frac{r}{2})} \frac{\Gamma_p(\frac{r}{2}, K) \Gamma_p(\alpha)}{\Gamma_p(\alpha + \frac{r}{2}, K)} C_K(Z_X) \\ &= \frac{|Z_X|^{\alpha + \frac{r}{2} - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\alpha + \frac{r}{2})} \frac{(\frac{r}{2})_K C_K(Z_X)}{(\alpha + \frac{r}{2})_K}. \end{aligned}$$

4. Extended Saigo Operators

Let

$$f(Z_X) = {}_2F_1(a, b; c; I - Z_X^{-\frac{1}{2}} Z_Y Z_X^{-\frac{1}{2}}) \phi(Z_Y), Z_X > Z_Y.$$

Here ${}_2F_1$ is a Gauss hypergeometric function of matrix argument $I - Z_X^{-\frac{1}{2}} Z_Y Z_X^{-\frac{1}{2}}$ and $\phi(Z_Y)$ is an arbitrary function so that

$$\int_{Z_X > Z_Y} |I - Z_X^{-\frac{1}{2}} Z_Y Z_X^{-\frac{1}{2}}|^{\alpha - \frac{p+1}{2}} C_K(I - Z_X^{-\frac{1}{2}} Z_Y Z_X^{-\frac{1}{2}}) \phi(Z_Y) dY < \infty.$$

Opening up the ${}_2F_1$ in terms of zonal polynomials and then substituting $U = A^{\frac{1}{2}} Y B A^{\frac{1}{2}}$, $V = U U'$ we have

$$\begin{aligned} &{}_0D_X^{-\alpha} [{}_2F_1(a, b; c; I - Z_X^{-\frac{1}{2}} Z_Y Z_X^{-\frac{1}{2}}) \phi(Z_Y)] \\ &= \sum_{k=0}^{\infty} \sum_K \frac{(a)_K (b)_K}{k! (c)_K} \frac{|Z_X|^{\alpha - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\alpha) \Gamma_p(\frac{r}{2})} \\ &\quad \int_{O < V < Z_X} |I - Z_X^{-\frac{1}{2}} V Z_X^{-\frac{1}{2}}|^{\alpha - \frac{p+1}{2}} C_K(I - Z_X^{-\frac{1}{2}} V Z_X^{-\frac{1}{2}}) \phi(V) |V|^{\frac{r}{2} - \frac{p+1}{2}} dV. \end{aligned} \quad (4.1)$$

Consider the special case $\phi(V) = |V|^\eta$ and then make the transformation $Z_X^{-\frac{1}{2}} V Z_X^{-\frac{1}{2}} = W$. Then the right side of (4.1) reduces to the following:

$$\sum_{k=0}^{\infty} \sum_K \frac{(a)_K (b)_K}{k! (c)_K} \frac{|Z_X|^{\alpha + \frac{r}{2} + \eta - \frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\alpha) \Gamma_p(\frac{r}{2})} \int_{O < W < I} |I - W|^{\alpha - \frac{p+1}{2}} |W|^{\eta + \frac{r}{2} - \frac{p+1}{2}} C_K(I - W) dW.$$

The integral part reduces to the following:

$$\begin{aligned}
& \int_{O < W < I} |W|^{\eta + \frac{r}{2} - \frac{p+1}{2}} |I - W|^{\alpha - \frac{p+1}{2}} C_K(I - W) dW \\
&= \int_{O < T < I} |T|^{\alpha - \frac{p+1}{2}} |I - T|^{\eta + \frac{r}{2} - \frac{p+1}{2}} C_K(T) dT \\
&= \frac{\Gamma_p(\alpha, K) \Gamma_p(\eta + \frac{r}{2})}{\Gamma_p(\alpha + \eta + \frac{r}{2}, K)} C_K(I) \\
&= \frac{\Gamma_p(\alpha) \Gamma_p(\eta + \frac{r}{2})}{\Gamma_p(\alpha + \eta + \frac{r}{2})} \frac{(\alpha)_K}{(\alpha + \eta + \frac{r}{2})_K} C_K(I).
\end{aligned}$$

Substituting back and denoting the left side by I_X we have

$$\begin{aligned}
I_X &= \frac{\pi^{\frac{r^2}{2}}}{|A|^{\frac{r}{2}} |B|^{\frac{p}{2}} \Gamma_p(\frac{r}{2})} \frac{\Gamma_p(\eta + \frac{r}{2})}{\Gamma_p(\alpha + \eta + \frac{r}{2})} |Z_X|^{\alpha + \eta + \frac{r}{2} - \frac{p+1}{2}} \\
&\quad \times {}_3F_2(a, b, \alpha; c, \alpha + \eta + \frac{r}{2}; I).
\end{aligned}$$

5. Some Statistical Considerations

Let $U_1 = A_1^{\frac{1}{2}} X_1 B_1 X_1' A_1^{\frac{1}{2}}$, $U_2 = A_2^{\frac{1}{2}} X_2 B_2 X_2' A_2^{\frac{1}{2}}$. Let X_1 and X_2 be independently and exponentially distributed random matrices where A_1 and A_2 are $p \times p$ real symmetric positive definite constant matrices, B_1 is $r_1 \times r_1$, $r_1 \geq p$ and B_2 is $r_2 \times r_2$, $r_2 \geq p$ constant positive definite matrices, X_1 is $p \times r_1$ and X_2 is $p \times r_2$ matrices of distinct real scalar random variables and the matrices be of full rank p , where a prime denotes the transpose and the square roots are the unique positive definite square roots of A_1 and A_2 respectively. Let the densities of X_1 and X_2 be denoted by $f_1(X_1)$ and $f_2(X_2)$ and the joint density, denoted by $f(X_1, X_2) = f_1(X_1)f_2(X_2)$ due to independence. Further, let

$$f_j(X_j) dX_j = c_j e^{-\text{tr}(A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}})} dX_j, j = 1, 2 \quad (5.1)$$

where c_j is the normalizing constant. Make the transformations $V_j = A_j^{\frac{1}{2}} X_j B_j^{\frac{1}{2}} \Rightarrow dX_j = |A_j|^{-\frac{r_j}{2}} |B_j|^{-\frac{p}{2}} dV_j$ where dX_j is the wedge product of all differentials in $X_k = (x_{ij}^{(k)})$ or $dX_k = \prod_{i=1}^p \prod_{j=1}^{r_k} \wedge dx_{ij}^{(k)}$, \wedge = wedge product. Let $W_j = V_j V_j'$ and integrate out over the Stiefel manifold. Then we have

$$dV_j = \frac{\pi^{\frac{r_j p}{2}}}{\Gamma_p(\frac{r_j}{2})} |W_j|^{\frac{r_j}{2} - \frac{p+1}{2}} dW_j \quad (5.2)$$

and $\text{tr}(\cdot)$ denotes the trace of (\cdot) . Hence the rectangular matrix X_j having the exponential density means W_j is having a real matrix-variate gamma density and further,

$$dX_j = \frac{\pi^{\frac{r_j p}{2}} |W_j|^{\frac{r_j}{2} - \frac{p+1}{2}}}{|A_j|^{\frac{r_j}{2}} |B_j|^{\frac{p}{2}} \Gamma_p(\frac{r_j}{2})} dW_j. \quad (5.3)$$

If the densities of V_j and W_j are denoted by $g_j(V_j)$ and $h_j(W_j)$ respectively then

$$h_j(W_j) dW_j = \frac{1}{\Gamma_p(\frac{r_j}{2})} |W_j|^{\frac{r_j}{2} - \frac{p+1}{2}} e^{-\text{tr}(W_j)} dW_j, W_j > O \quad (5.4)$$

$$g_j(V_j) dV_j = \frac{1}{\pi^{\frac{r_j p}{2}}} e^{-\text{tr}(V_j V_j')} dV_j \quad (5.5)$$

$$f_j(X_j) dX_j = \frac{|A_j|^{\frac{r_j}{2}} |B_j|^{\frac{p}{2}}}{\pi^{\frac{r_j p}{2}}} e^{-\text{tr}(A_j^{\frac{1}{2}} X_j B_j X_j' A_j^{\frac{1}{2}})} dX_j. \quad (5.6)$$

5.1. Density of the sum

Let us examine the density of the sum $U_1 + U_1$. The joint density of X_1 and X_2 , denoted by $f(X_1, X_2)$, is given by

$$f(X_1, X_2)dX_1 \wedge dX_2 = f_1(X_1)f_2(X_2)dX_1 \wedge dX_2.$$

Make the transformations $V_j = A_j^{\frac{1}{2}}X_jB_j^{\frac{1}{2}}, W_j = V_jV_j'$ then

$$\begin{aligned} f(X_1, X_2)dX_1 \wedge dX_2 &= f_1(V_1V_1')f_2(V_2V_2')\left\{\prod_{j=1}^2 |A_j|^{-\frac{r_j}{2}}|B_j|^{-\frac{p}{2}}\right\}dV_1 \wedge dV_2 \\ &= f_1(W_1)f_2(W_2)\left\{\prod_{j=1}^2 |A_j|^{-\frac{r_j}{2}}|B_j|^{-\frac{p}{2}}\frac{\pi^{\frac{r_j}{2}}}{\Gamma_p(\frac{r_j}{2})}|W_j|^{\frac{r_j}{2}-\frac{p+1}{2}}\right\}dW_1 \wedge dW_2. \end{aligned}$$

Then $U_1 + U_2 = W_1 + W_2$. Make the transformation $U = W_1 + W_2$ and $V = W_1$, the Jacobian is unity and $W_1 = V, W_2 = U - V$ and the integration is over the positive definite matrices U and V with $U - V > O$. If X_1 and X_2 are independently distributed as in (5.1) then denoting the marginal density of U by $f^*(U)$ we have

$$f^*(U)dU = \int_{U>V>O} f_1(U - V)f_2(V)dV \quad (5.7)$$

$$= \frac{1}{\Gamma_p(\frac{r_1}{2})\Gamma_p(\frac{r_2}{2})}|U - V|^{\frac{r_1}{2}-\frac{p+1}{2}}|V|^{\frac{r_2}{2}-\frac{p+1}{2}}e^{-\text{tr}(U-V)-\text{tr}(V)}dV \wedge dV. \quad (5.8)$$

Make the transformation $Z = U^{-\frac{1}{2}}VU^{-\frac{1}{2}}$ for fixed U , then

$$\begin{aligned} f^*(U)dU &= \frac{|U|^{\frac{r_1+r_2}{2}-\frac{p+1}{2}}e^{-\text{tr}(U)}}{\Gamma_p(\frac{r_1}{2})\Gamma_p(\frac{r_2}{2})}\int_{Z>O}|Z|^{\frac{r_2}{2}-\frac{p+1}{2}}|I - Z|^{\frac{r_1}{2}-\frac{p+1}{2}}dZ \wedge dU \\ &= \frac{|U|^{\frac{r_1+r_2}{2}-\frac{p+1}{2}}e^{-\text{tr}(U)}}{\Gamma_p(\frac{r_1+r_2}{2})}dU. \end{aligned}$$

Thus the sum U is real matrix-variate gamma distributed. Note that the integral in (5.7) is Riemann-Liouville left-sided fractional integral for the power function or where the arbitrary function is of the form $f(T) = |T|^{\frac{r_2}{2}-\frac{p+1}{2}}$.

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